A new algorithmic framework for enumerating commutable set properties

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Set systems

Definitions

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An **independence system** is a set system that is closed under subsets, so that if $A \in \mathcal{F}$ then any $B \subseteq A$ also belongs to $\mathcal{F}$.

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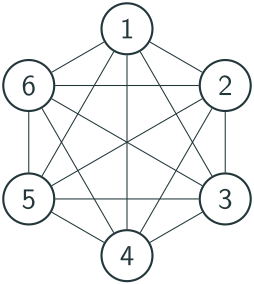

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A **strongly accessible set system** is a set system that satisfies the following property: given $A, B \in \mathcal{F}$, if $A \subsetneq B$ then there is an $a \in B \setminus A$ such that $A \cup \{a\} \in \mathcal{F}$.

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Definition
A **commutable set system** is a strongly accessible set system that satisfies the following property: given $A, B, C, D \in \mathcal{F}$, if $A \subset B \cap C$, $B \cup C \subset D$ then $B \cup C \in \mathcal{F}$.

Example: black-connected cliques

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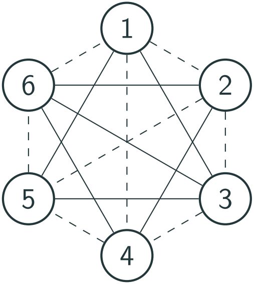
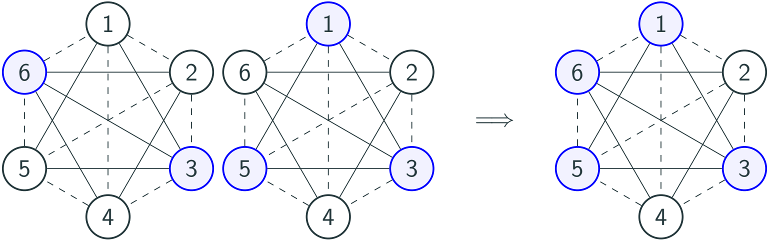


The “commutable” property for black-connected cliques



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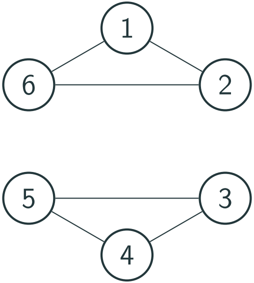

$k = 4$

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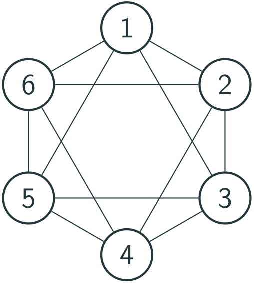

$k = 2$

Our goal

The objective of our framework is to **enumerate all the maximal sets** (by inclusion) in a commutable set system.

Reverse search

Definition
Let $\mathcal{F}$ be a strongly accessible set system. We say that *parent* is a **parent function** if it has the following properties:

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- $\mathit{parent}(S)$ is another maximal set in $\mathcal{F}$, for all the maximal solutions S for which it is defined (so for all the S that are not a *root*).
- There is an order $\prec$ such that $\mathit{parent}(S) \prec S$ for all the S that are not a *root*.

Definition
If *parent* is a parent function, we define $\mathit{children}(S)$ for a maximal solution S to be the set of all maximal solutions C such that $\mathit{parent}(C) = S$.

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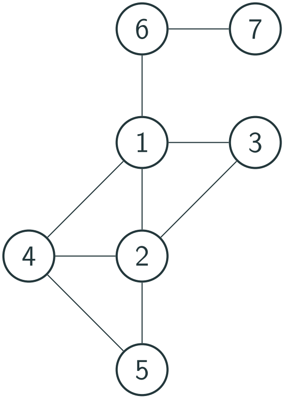
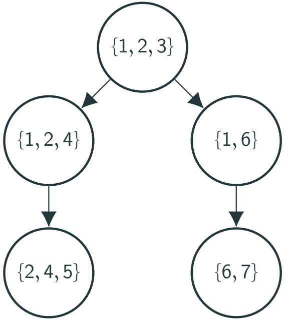
Theorem
Let G be the directed graph that has the maximal solutions of $\mathcal{F}$ as nodes, and has an outgoing edge from S to any solution in $\mathit{children}(S)$. Then G is a directed forest rooted in the solutions that are roots.

On the right is a possible reverse search graph for the cliques in the graph on the left.





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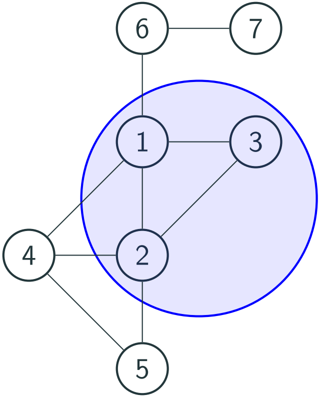

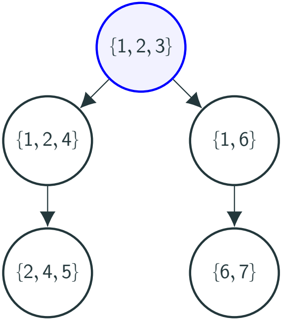

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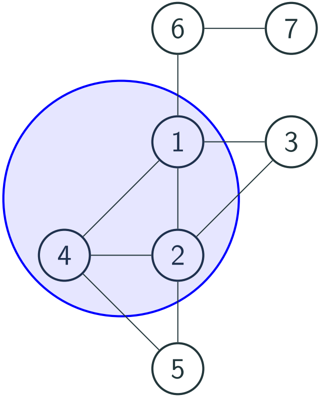

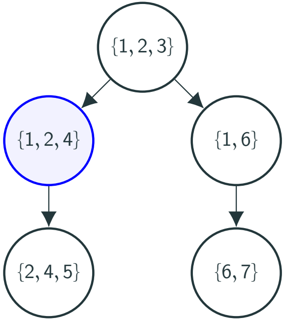

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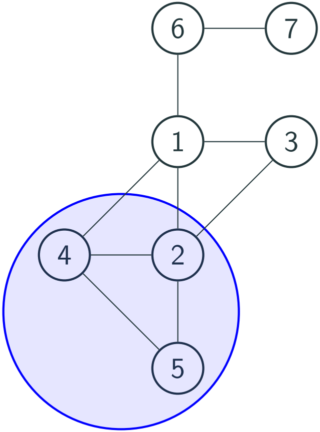

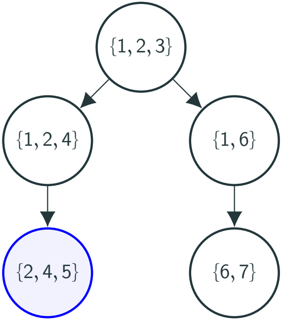

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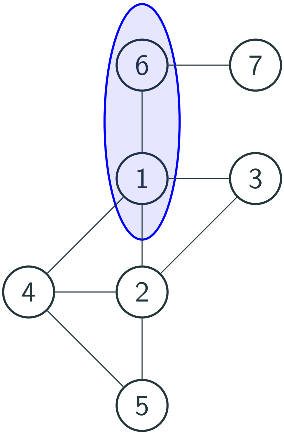

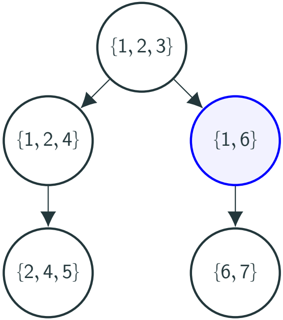

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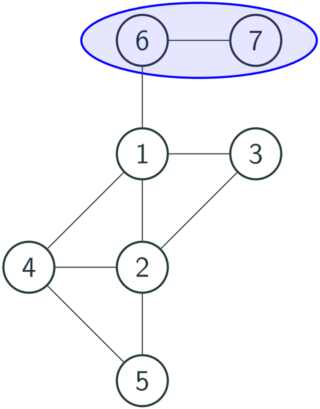

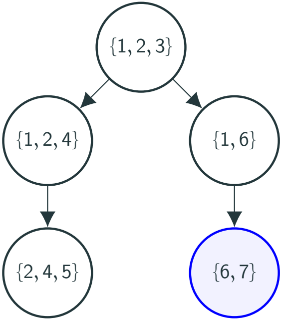

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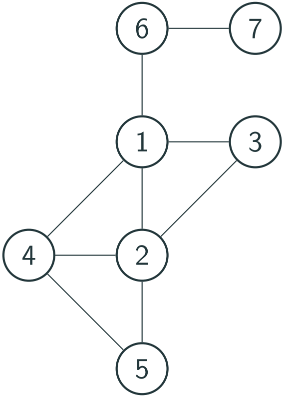
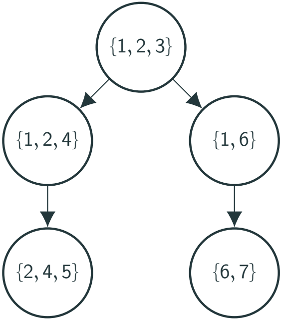
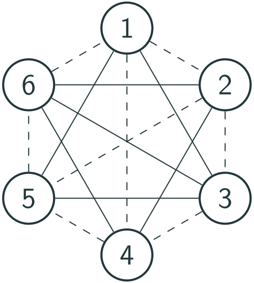




parent and *children* for commutable set systems

Definition
Given a commutable set system $\mathcal{F}$ and one of its feasible sets S, we say that s is a **seed** of S if $s \in S$ and $\{s\} \in \mathcal{F}$. The **canonical seed** of S, denoted by $seed(S)$, is the smallest possible seed according to the ordering of the elements of E.

Example of these definitions



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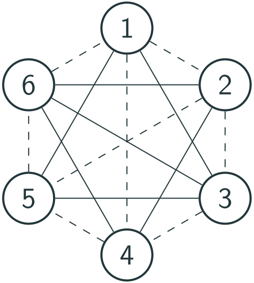

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- if there is no such subset, we say that the level of v is ∞.

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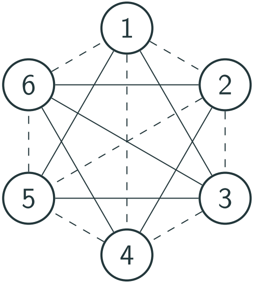

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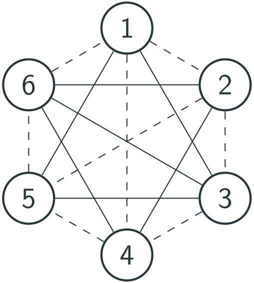

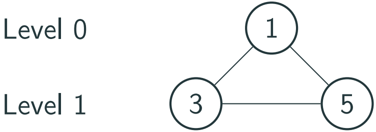

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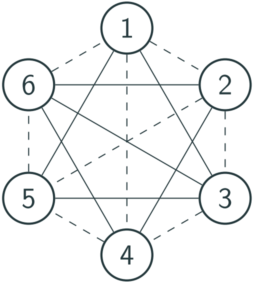

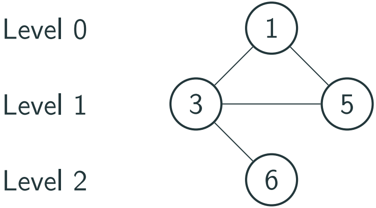

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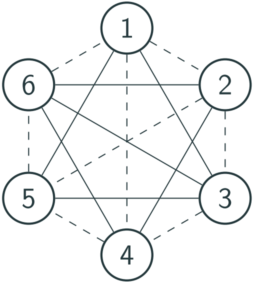

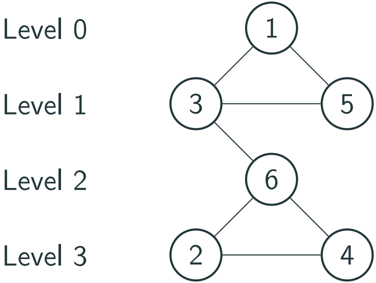

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Level order

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The **level order** $\prec$ between any two solutions P, Q of a commutable set system $\mathcal{F}$ is defined as follows:

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- Let lQ be defined in the same way.
- We say that $P \prec Q$ if and only if lP is smaller than lQ according to lexicographical order.

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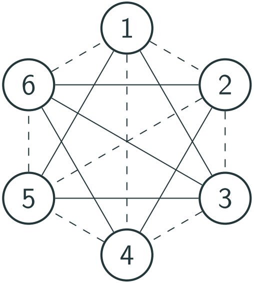

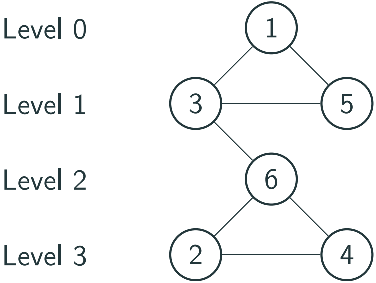

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$$IS = [(0,1),(1,3),(1,5),(2,6),(3,2),(3,4)]$$

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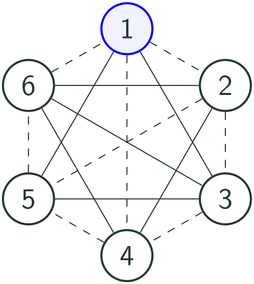

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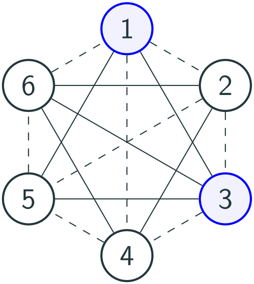

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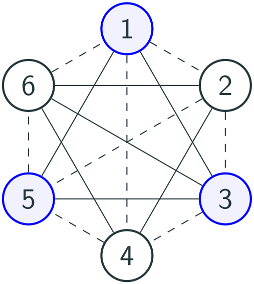

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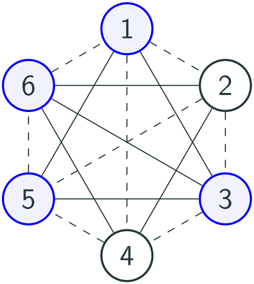

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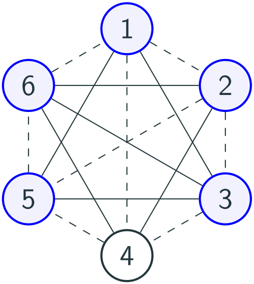

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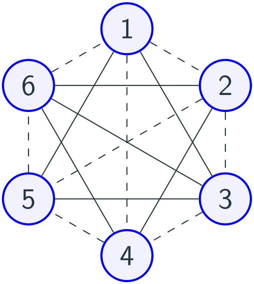

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Definition
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We can prove that *parent*$(S) \prec S$.

Definition
Given a strongly accessible set system $\mathcal{F}$ on E, a maximal feasible set S and an element $v \in E \setminus S$, the **restricted problem** $\mathcal{P}(S, v)$ is the problem of enumerating all the maximal elements of the family

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Theorem
Given a maximal solution C such that $parent(C) = S$, $core(C)$ is the prefix ending just before $parind(C)$ of a solution of $\mathcal{P}(S, parind(C))$.

Applications

Cliques and black-connected cliques

In both cases, the restricted problem is easy and has at most one solution. Let q be the maximum size of a black-connected clique. Then

Theorem
All the maximal black-connected cliques in a graph G may be enumerated in $O(q^5\Delta_b^2)$ time per solution, using only $O(q)$ extra memory (other than the memory used to store G).

Connected k-plexes

Lemma
Assuming k to be a constant, if S is a k-plex and v is a node in $V \setminus S$, then there are at most $1 + f(k)|S|^{k-1}$ maximal k-plexes in $S \cup \{v\}$, with $f(k) = (k-1)^{2k}$ for $k > 1$ and $f(1) = 1$. Moreover, they can be computed in $O(kf(k)|S|^k)$ time using only $O(kq)$ memory.

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Conclusions

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Any questions?